ON THE INTERGRABILITY OF THE MARTINGALE SQUARE FUNCTION

BY
BURGESS DAVIS

ABSTRACT

Let $f = (f_1, f_2, ..)$ be a martingale. It is proved that the L_1 norms of $\sup_n |f_n|$ and of $(\Sigma(f_n - f_{n-1})^2)^{\frac{1}{2}}$ are equivalent. This result completes results of D. L. Burkholder and R. F. Gundy on operators on martingales.

In this paper we prove the following:

THEOREM 1. *Let $f = (f_1, f_2, \cdots)$ be a martingale, where $f_n = \sum_{k=1}^{n} d_k$, $n \geqq 1$. Define $f^* = \sup_n |f_n|$ and $S(f) = (\sum_{k=1}^{\infty} d_k^2)^{\frac{1}{2}}$. Then there are two positive numbers c, C such that*

$$c \| S(f) \|_1 \leqq \| f^* \|_1 \leqq C \| S(f) \|_1 . \tag{1}$$

This inequality completes a ine of investigation begun by D. L. Burkholder, who proved that there exist constant c_p, C_p such that

$$c_p \| S(f) \|_p \leqq \| f^* \|_p \leqq C_p \| S(f) \|_p \tag{2}$$

for $1 < p < \infty$. (See theorem 9 of [2] and p. 317 of [3].) More recently, in [1], inequality (2) has been extended to the entire interval $0 < p < \infty$ for martingales that satisfy a certain regularity condition, and it is shown that inequality (2) fails to hold without this regularity condition when $0 < p < 1$. Theorem 1 settles the question for the only remaining case, $p = 1$.

PROOF OF THEOREM 1. Define f_n^* and $S_n(f)$ to be 0 if $n = 0$, $\max_{i \leqq n} |f_i|$ and $(\sum_{i=1}^{n} d_i^2)^{\frac{1}{2}}$ respectively if $n > 0$.

Since

$$\| S(f) \|_2^2 = \lim \| S_n(f) \|_2^2 = \lim \| f_n \|_2^2 = \| f \|_2^2,$$

the most immediate connections between f^* and $S(f)$ are

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(3) $$P(S(f) > \lambda) \leqq \| f^* \| \|_2^2/\lambda^2$$

(4) $$P(f^* > \lambda) \leqq \| S(f) \|_2^2/\lambda^2,$$

(4) following by a result of Doob (3.4′, p. 314 of [3]) applied to the sub-martingale $(f_n^2, n \geqq 1)$.

Now let T be one of the operators $f \to f^*$, $f \to S(f)$ and let H be the other. Theorem 1 will be proved by showing that if f is a martingale then $\| H(f) \|_1 \leqq 130 \| T(f) \|_1$, which is sufficient to establish the double inequality (1) since the roles of H and T can be reversed.

Define

$$A_n = \{|d_n| > 2d^*_{n-1}\}$$

$$\Delta_n = E(d_n I(A_n) \,|\, f_i, i < n),$$

and

$$a_n = d_n I(A_n) - \Delta_n.$$

Then a is a martingale difference sequence. Let $h_n = \Sigma_{i=1}^n a_i$. Since $|d_n| < 2(d_n^* - d^*_{n-1})$ where $|d_n| > 2d^*_{n-1}$, we have

$$\Sigma \,|d_n I(A_n)| \leqq 2 \;\Sigma\, (d_n^* - d^*_{n-1}) = 2d^* \leqq 4T(f),$$

and so

(5) $$\| \;\Sigma\, |\Delta_n| \;\|_1 \leqq 4 \| T(f) \|_1$$

Define $g = (f - h)$. Then

$$\| H(f) \|_1 \leqq \| H(g) \|_1 + \| H(h) \|_1$$
$$\leqq \| H(g) \|_1 + \| \;\Sigma\, |a_n| \,\|_1$$
$$\leqq \| H(g) \|_1 + 8 \| T(f) \|_1.$$

We now proceed to estimate $\| H(g) \|_1$. Let $t = t(\lambda)$ be the stopping time

$$\inf\{n : \max(T_n(f), T_n(g), \Delta_{n+1}) > \lambda\}.$$

Then $|g_t - g_{t-1}| = |d_t I(|d_t| \leqq 2d^*_{t-1}) + \Delta_t| \leqq 5\lambda$, since $|\Delta_t| \leqq \lambda$ and $d^*_{t-1} \leqq 2|T_{t-1}(f)| \leqq 2\lambda$, so $T_t(g) \leqq T_{t-1}(g) + |g_t - g_{t-1}| \leqq 6\lambda$. In addition $T_t(g) \leqq T(g) \leqq T(f) + T(h)$.

$$
\begin{aligned}
\text{Now } P(H(g) > \lambda) &\leqq P(t < \infty) + P(t = \infty,\ H(g) > \lambda) \\
&\leqq P(t < \infty) + P(H_t(g) > \lambda), \text{ and}
\end{aligned}
$$

$$
\begin{aligned}
\int_0^\infty P(t < \infty)\, d\lambda &\leqq \int_0^\infty P(T(f) > \lambda) d\lambda + \int_0^\infty P(T(g) > \lambda) d\lambda + \int_0^\infty P(\Sigma |\Delta_n| > \lambda) d\lambda \\
&= \| T(f) \|_1 + \| T(g) \|_1 + \| \ \Sigma \ |\Delta_n| \ \|_1 \\
&\leqq 14 \| T(f) \|_1, \text{ while}
\end{aligned}
$$

$$
\begin{aligned}
\int_0^\infty P(H_t(g) > \lambda) d\lambda &\leqq \int_0^\infty \| T_t(g) \|_2^2 d\lambda/\lambda^2 \text{ (by (3) or (4))} \\
&\leqq \int_0^\infty \| \min(T(f) + T(h), 6\lambda) \|_2^2 d\lambda/\lambda^2 \\
&\leqq \int_0^\infty \| (T(f) + T(h)) I(T(f) + T(h) \leqq 6\lambda) \|_2^2 d\lambda/\lambda^2 \\
&\quad + \int_0^\infty P(T(f) + T(h) > 6\lambda)(6\lambda)^2 d\lambda/\lambda^2 \\
&= \int_0^\infty \| (T(f) + T(h)) I(T(f) + T(h) \leqq 6\lambda) \|_2^2 d\lambda/\lambda^2 \\
&\quad + 6 \| T(f) + T(h) \|_1 \\
&= \int_\Omega (T(f) + T(h))^2 \int_{(T(f)+T(h))/6}^\infty d\lambda/\lambda^2 + 6 \| T(f) + T(h) \|_1 \\
&= 12 \| T(f) + T(h) \|_1 \\
&\leqq 108 \| T(f) \|_1
\end{aligned}
$$

Thus $\| H(g) \|_1 = \int_0^\infty P(H(g) > \lambda) d\lambda \leqq 122 \| T(f) \|_1$, so $\| H(f) \|_1 \leqq 130 \| T(f) \|_1$ completing the proof of Theorem 1.

We now indicate an extension of Theorem 1 to a class of more general operators. A matrix operator is defined to be an operator which can be expressed in the form

$$
M(f) = \left[\sum_{j=1}^{\infty} (\limsup_{n \to \infty} \left| \sum_{k=1}^{n} a_{jk} d_k \right|)^2 \right]^{\frac{1}{2}}
$$

where $(a_{jk}\ \ 1 \leqq j < \infty,\ 1 \leqq k < \infty)$ is a matrix of real numbers such that $c \leqq \sum_{j=1}^{\infty} a_{jk}^2 \leqq C$, $k \geqq 1$, c and C positive numbers. $M_n(f)$ denotes $M(f^n)$, where f^n is f stopped at time n, and $M^*(f) = \sup_n M_n(f)$. Such operators are discussed in [1], where it is shown that if M and N are two matrix operators then there is a number $c(M,N,p)$ such that if f is a martingale then

$$\| M^*(f) \|_p \leqq c(M,N,p) \| N^*(f) \|_p, \quad p > 1$$

and that for martingales satisfying certain regularity conditions this inequality may be extended to the range $0 < p < \infty$. Richard F. Gundy has observed that with minor changes the proof of Theorem 1 proves the following theorem.

THEOREM 2. *Let M and N be matrix operators. Then there is a constant $c(M,N)$ such that if f is a martingale then*

$$\| M^* f \|_1 \leqq c(M,N) \| N^* f \|_1 .$$

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RUTGERS UNIVERSITY